$\Rightarrow$ and $\Leftarrow$ $\quad$ $f(x) < f(c)$ for all $x \in D$, $x \neq c$

$c \in D$ is a (strict) maximum point of f $\Leftrightarrow$ $f(x) \leq f(c)$ for all $x \in D$

point should be in the domain D $\quad$ generic

$\Leftrightarrow$ characterization (if and only if statement)

$d \in D$ is a minimum point of f $\Leftrightarrow$ $f(x) \geq f(d)$ for all $x \in D$

- No calculus involved.
- No assumption about f being continuous.
- No assumption about f being differentiable. → first derivative exists and is well-defined.
- It does not tell you / does not give a recipe to find max/min.

	c		$f(c)$
(maximizer)	maximum point	vs	maximum value
(minimizer)	minimum point	vs	minimum value
	d		$f(d)$
	optimal / extreme points		optimal / extreme values.

Example.
$f(x) = 3$ $\quad$ $x = 1$ is a maximum pt of f. why? Because $3 = f(x) \leq f(1) = 3$ for all $x \in \mathbb{R}$

$x \in \mathbb{R}$ (the set of real numbers) → D

but

$x = 1$ is not a strict maximum point of f.
because $f(x)$ is not less than $f(1) = 3$ for all $x \neq 1$.

Example.
$f(x) = x^2$ $\quad$ Find minimum points of f.

$x = 0$ is a minimum point of f.

Convert into a maximization problem.
$g(x) = -f(x) = -x^2$ $\Rightarrow$ $x = 0$ is a maximum point of f.

Section 8.1 Example 1
You will see commonly used functions. If you know their behavior, you can choose not to use calculus to find extreme points.

$(x-2)^2$

$f(x) = 3-(x-2)^2, \quad x \in \mathbb{R}$

$x=2$ is a max pt of f.

no minimum pts of f

$(x-2)^2 \geq 0$ for all $x \in \mathbb{R}$

$\Rightarrow (-1)(x-2)^2 \leq 0$ for all $x \in \mathbb{R}$ $\quad +3 \quad +3$

$\Rightarrow \underbrace{\underbrace{-(x-2)^2+3}_{3-(x-2)^2}}_{f(x)} \leq 3$

$-4+3=-1$

(algebra, applying properties of inequalities)

$-(x-2)^2 \leq 0$

$-(x-2)^2+3 \overset{?}{\leq} 0+3$

What we have shown is that $f(x) \leq 3$ for all $x \in \mathbb{R}$.

Go back to def of max:

$c \in D$ is a max pt of $f \Leftrightarrow f(x) \leq f(c)$ for all $x \in D$

$2 \in \mathbb{R}$ is a max pt of $f \Leftrightarrow f(x) \leq 3$ for all $x \in \mathbb{R}$

$f(2)=3$

- How to write arguments / proofs
- Seeing many ways of doing the same thing

inequalities

$a \leq b$

$a+c \leq b+c$

$-(x-2)^2 \leq 0$

$-(x-2)^2+3 \leq 0+3$

$f(x) = 3-(x-2)^2$

$\lim_{x \to \infty} f(x) = ?$

$x \to -\infty$

$x \to \infty \Rightarrow (x-2)^2 \to \infty$

$\Rightarrow -(x-2)^2 \to -\infty$

$\Rightarrow 3-(x-2)^2 \to -\infty$

Similarly for $x \to -\infty$

$\Rightarrow$ no minimum pt of f

$c(t) = \dfrac{t}{t^2+4}$

(b) $\quad D = [5, +\infty)$

included

not a number

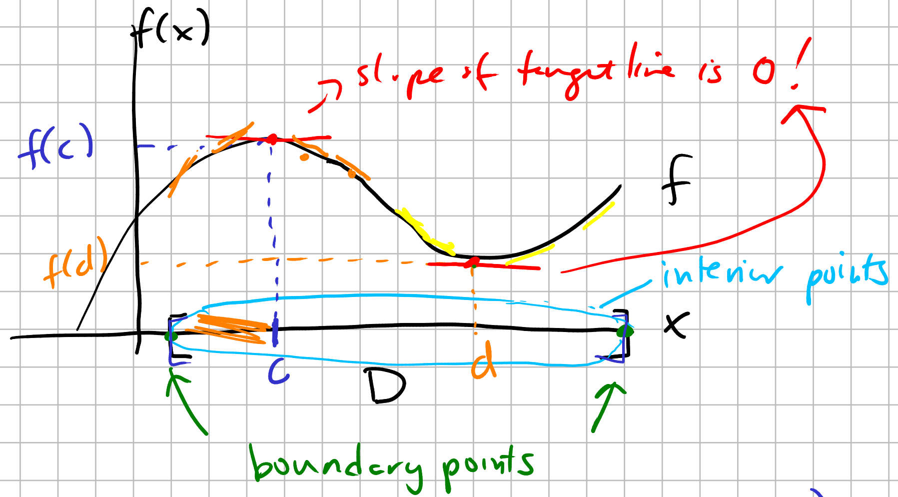
f differentiable (means that first derivative of f exists and is well-defined)

c is an interior point of D ("inside" D, not at the boundaries)

c is a max/min pt of f

$\boxed{\Rightarrow}$ $f'(c)=0$

Geometry behind ↑



Here c is a max point of f (interior of D)

Here d is a min point of f (interior of D)

- Finding candidates for extreme points (interior of D)

  solving for x $\boxed{f'(x)=0}$ FOC

- Use First Derivative Test to determine whether you have max/min.

Section 8.2 Example 1

$$c(t)=\frac{t}{t^2+4}$$

Find max/min pts of $c(t)$ or what values of t will maximize/minimize the function $c(t)$?

numerator (function of t)

denominator (function of t)

↳ pattern of quotient/ratio

Apply quotient rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$$\frac{\text{Lo d hi} - \text{hi d Lo}}{\text{Lo}^2}$$

Step 1

$$c'(t)=\frac{(t^2+4)\frac{d}{dt}[t]-t\frac{d}{dt}[t^2+4]}{(t^2+4)^2}$$

$$=\frac{(t^2+4)(1)-t(2t)}{(t^2+4)^2}$$

$$=\frac{t^2+4-2t^2}{(t^2+4)^2}$$

$$=\frac{-t^2+4}{(t^2+4)^2}$$

$$\frac{d}{dt}[t]$$

$$\frac{d}{dx}[c] = 0$$

c is a constant

$$\frac{d}{dx}[x^n] = n x^{n-1}$$

power rule

n is a constant

$$\frac{d}{dt}[t^2+4]$$

$$= \frac{d}{dt}[t^2] + \frac{d}{dt}[4]$$

$$\frac{d}{dx}[f(x)+g(x)]$$

$$= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Step 2 $\quad c'(t) = 0$

what values of t will make

$$\frac{-t^2+4}{(t^2+4)^2} = 0 \Rightarrow \text{when } -t^2+4=0.$$

$$\Rightarrow -(t^2-4) = 0$$

$$\Rightarrow -(t-2)(t+2) = 0$$

either $t-2=0$ or $t+2=0$

$t=2$ $\qquad t=-2$

So, $t=-2$, $t=2$ are stationary points of $c(t)$.

NOTE the example states $t \geq 0$, so we can already rule out $t=-2$.

Step 3 Apply first derivative test. (Signs of $c'(t)$)

When is it positive?

When is it negative?

t: 0, 2

$$c'(t) = \frac{-(t-2)(t+2)}{(t^2+4)^2} \quad - \frac{(-)(+)}{} > 0$$

when $0 \leq t < 2$

when $0 \leq t < 2$

always positive

When $0 \leq t < 2$, $t-2$, $t+2$

When $0 \leq t < 2$, $c'(t) > 0$

Complete the argument here.... When $t > 2$, $c'(t) < 0$

$\Rightarrow$ $t=2$ is a maximum point of $c(t)$ by the first deriv test.