Section 8.2 Exercise 9

$a_1 - , a_n$ (constants do not depend on x)

$$d(x) = (x-a_1)^2 + (x-a_2)^2 + \dots + (x-a_n)^2$$

$$d'(x) = 2(x-a_1)(1) + 2(x-a_2)(1) + \dots + 2(x-a_n)(1)$$

$$= 2[x - a_1 + x - a_2 + \dots + x - a_n] \qquad \underbrace{x + x + \dots + x}_{n \text{ times}} = nx$$

$$= 2[nx - a_1 - a_2 - \dots - a_n] = 2n\left[x - \frac{a_1 + a_2 + \dots + a_n}{n}\right]$$

candidate extreme pts:

$$d'(x) = 0 \qquad \frac{2(nx - a_1 - a_2 - \dots - a_n)}{2} = \frac{0}{2}$$

$$nx - a_1 - a_2 - \dots - a_n = 0$$

$$nx = a_1 + a_2 + \dots + a_n$$

$$x = \frac{a_1 + a_2 + \dots + a_n}{n}$$

If you want to use first deriv test,

$x < \frac{a_1 + \dots + a_n}{n}$ Is $d'(x) > 0$ or $d'(x) < 0$?

$x > \frac{a_1 + \dots + a_n}{n}$ Is $d'(x) > 0$ or $d'(x) < 0$?

$$d'(x) = 2(\boxed{nx} - a_1 - \dots - a_n) = \underbrace{2n\left(x - \frac{a_1 + \dots + a_n}{n}\right)}_{\text{a form that could allow us to say what is the sign of } d'(x)}$$

Check whether $d(x)$ is convex/concave?

$$d''(x) = \underbrace{2(n)}_{\text{does not depend on x at all.}} > 0 \qquad \text{because } n > 0$$

Thm 8.2.2 Since $d(x)$ is convex over all $x \in \mathbb{R}$, and $\frac{a_1 + \dots + a_n}{n} \in \mathbb{R}$ stationary pt., by Thm 8.2.2, $\frac{a_1 + a_2 + \dots + a_n}{n}$ is a minimum of $d(x)$.

Section 8.2 Exercise 10

$$f(x) = I_0 + kx + Ae^{-\alpha x}$$

(a) $f'(x) = k + Ae^{-\alpha x}(-\alpha) = k - A\alpha e^{-\alpha x}$

candidate extreme points

$$k - A\alpha e^{-\alpha x} = 0$$

$$k = A\alpha e^{-\alpha x}$$

$$\frac{k}{A\alpha} = e^{-\alpha x}$$

$$\ln\left(\frac{k}{A\alpha}\right) = \ln e^{-\alpha x}$$

$$\ln\left(\frac{k}{A\alpha}\right) = -\alpha x$$

$$\boxed{-\underbrace{\frac{1}{\alpha}}_{>0}\underbrace{\ln\left(\frac{k}{A\alpha}\right)}_{<0} = x}$$

It might not be obvious, but $x > 0$.

$$f'(x) = \boxed{k} - A\alpha e^{-\alpha x}$$

$$f''(x) = -A\alpha \cdot e^{-\alpha x}(-\alpha)$$

$$= \underbrace{A}_{>0}\underbrace{\alpha^2}_{>0}\underbrace{e^{-\alpha x}}_{>0} > 0$$

for any $x \geq 0$

$$A\alpha > k$$

$$\ln A\alpha > \ln k$$

$$0 > \underbrace{\ln k - \ln A\alpha}$$

$$0 > \ln\left(\frac{k}{A\alpha}\right)$$

$\ln(\cdot)$ is increasing function

$\ln x$ is defined when $x > 0$, $\frac{d}{dx}[\ln x] = \frac{1}{x} > 0$

$$\ln x^p = p \ln x$$

$x > 0$

why.

$$x = -\frac{1}{\alpha}\ln\left(\frac{k}{A\alpha}\right) = \frac{1}{\alpha}\ln\left(\frac{k}{A\alpha}\right)^{-1} = \frac{1}{\alpha}\ln\left(\frac{A\alpha}{k}\right)$$

$$x_0 = \frac{1}{\alpha}\ln\left(\left(\frac{\alpha p_0 V}{k}\right)\left(1 + \frac{100}{\delta}\right)\right) = \frac{1}{\alpha}\left[\ln\left(\frac{\alpha p_0 V}{k}\right) + \underbrace{\ln\left(1 + \frac{100}{\delta}\right)}_{\neq \ln 1 + \ln \frac{100}{\delta}}\right]$$

$$= \frac{1}{\alpha}\left[\ln \alpha + \ln p_0 + \ln V - \ln k + \ln\left(1 + \frac{100}{\delta}\right)\right]$$

$$\frac{\partial x_0}{\partial p_0} = \frac{1}{\alpha} > 0$$

Section 8.3 (a)

$$P(Q) = a - Q$$

$$\text{costs} = kQ$$

$0 < \boxed{k < a}$

refer to Example 3

$$Q^* = \frac{a-k}{2} > 0$$

concave in Q

$$\pi(Q) = QP(Q) - kQ$$

$$= Q(a-Q) - kQ = aQ - Q^2 - kQ$$

$$\pi''(Q) = -2 < 0$$

regardless of what Q is

$$\pi'(Q) = a - 2Q - k$$

candidate extreme pts

$$a - 2Q - k = 0$$

$$a - 2Q - k = 0$$

$$a - k = 2Q$$

$$\frac{a-k}{2} = Q^*$$

$$\pi(Q) = aQ - Q^2 - kQ = \underbrace{(a-k)}Q - Q^2$$

Section 8.3 (c) $Q^* = \frac{a-k}{2}$ $\longrightarrow$ $\hat{Q} = a-k$

$$\pi(Q) = \underset{\text{revenues}}{QP(Q)} \underset{-\text{costs}}{- kQ} \underset{+\text{ subsidy}}{+ sQ} = Q(a-Q) - kQ + sQ$$

$$= aQ - Q^2 - kQ + sQ$$

$$\pi'(Q) = (a-k+s) - 2Q$$

$\pi''(Q) = -2 < 0$
regardless of Q

$$a-k+s-2Q = 0$$

$$\frac{a-k+s}{2} = Q^{**} \text{ (profit maximizing quantity under subsidy)}$$

$$\frac{a-k+s}{2} = a-k$$

(what the gov desires)

$$a-k+s = 2(a-k)$$

$$\underline{s} = 2a-2k-a+k = \underline{\underline{a-k}}$$

Sec 8.4 #3

$-1, 2$ endpoints included in $[-1,2]$ $\longrightarrow$ candidate extreme points

$$g(x) = \frac{1}{5}\left(e^{x^2} + e^{2-x^2}\right)$$

$$g'(x) = \frac{1}{5}\left(e^{x^2}(2x) + e^{2-x^2}(-2x)\right)$$

will never be undefined
(If there were an x so that g' will be undefined, then it is also a candidate extreme pt)

g(x) defined

$$\boxed{\frac{1}{5}}\left(e^{x^2}(2x) + e^{2-x^2}(-2x)\right) = 0$$

$$e^{x^2}(2x) + e^{2-x^2}(-2x) = 0$$

(If it were

$$2x\left(e^{x^2} + e^{2-x^2}\right) = 0$$

$\neq 0$

$$2x\left(e^{x^2} - e^{2-x^2}\right) = 0$$

$2x = 0$ $\quad$ $\sim$ $\quad$ $e^{x^2} - e^{2-x^2} = 0$

$x = 0$

$$e^{x^2} = e^{2-x^2}$$

$$\ln e^{x^2} = \ln e^{2-x^2}$$

$\ln e = 1$

$$x^2 = 2 - x^2$$

$$2x^2 - 2 = 0$$

$$x^2 = 1 \Longrightarrow x = \pm 1$$

candidate extreme points $x = 0, +1, -1, 2$

g is continuous on a closed & bounded interval

By Extreme Value Thm, there exists a max / a min.

$g(0) = \frac{1}{5}(1+e^2) =$ $g(2) = \frac{1}{5}(e^4+1) =$

$g(1) = \frac{1}{5}(e+e) = \frac{2}{5}e =$

$g(-1) = \frac{1}{5}(e+e) = \frac{2}{5}e =$

$g(x) = \frac{1}{5}\left(e^{x^2} + e^{2-x^2}\right)$

If you don't have a calc.

$\frac{2}{5}e < \frac{1}{5}(1+e^2) < \boxed{\frac{1}{5}(1+\boxed{e^4})}$

2.7

$-1, 1 \to$ global minima.

$2 \to$ global maximum

$e \approx 2.7$ $\quad e^2 \approx 9$

$\frac{2}{5}e \approx 1.08$