unconstrained

Ch 8 optimization functions of one variable

Ch 13 unconstrained " functions of two or more variables } some exercises / examples have constraints.

Ch 14 " " . .. .

constrained

Section 13.2 Ex 7 constraint that $4x+2y-z=5$

Section 13.5 Ex 3 constraints that $0 \le x \le 5$, $0 \le y \le 3$, $-x+2y \le 2$

Types of constraints — equality — inequality

possible to have multiple constraints, possible to have a mix of types.

economics → typically equality constraints show up.

→ in situations where economic agents face uncertainty, inequality constraints typically show up.

$$\underline{\mathcal{L}(x,y)} = f(x,y) - \lambda(g(x,y)-c)$$

lambda = Lagrange multiplier

$$\max/\min\ f(x,y) \ \text{s.t.}\ \boxed{g(x,y)=c}$$

subject to

$$\mathcal{L} = f(x,y) - \lambda(g(x,y)-c)$$

$f(x,y)$

$$\begin{cases} \max\ xy \\ \text{subject to}\ \underbrace{2x+y}_{g(x,y)} = \underbrace{100}_{c} \end{cases}$$

$$\mathcal{L} = xy - \lambda(2x+y-100)$$

$$\mathcal{L}'_x = y - 2\lambda$$

$$\mathcal{L}'_y = x - \lambda$$

$$\begin{cases} y - 2\lambda = 0 \\ x - \lambda = 0 \end{cases}$$

$$2x+y=100$$

$$y = 2\lambda \Rightarrow \frac{y}{2} = \lambda$$

$$x = \lambda$$

$$x = \frac{y}{2}$$

$$\Rightarrow 2x = y$$

$$\left.\begin{matrix} 2x = y \\ 2x+y=100 \end{matrix}\right\} \quad y+y=100 \Rightarrow 2y=100$$

$$\Rightarrow y = 50 \Rightarrow x = y/2 = 25$$

$$\lambda = y/2 \ \text{or}\ \lambda = x \Rightarrow \lambda = 25$$

Section 14.1 Example 3

$$\left.\begin{aligned} x &= \frac{a}{a+b}\frac{m}{p} \\ y &= \frac{b}{a+b}\frac{m}{q} \end{aligned}\right\}$$

solutions to max utility s.t. budget constraint

$\to$ obtained consumer demand function

$a, b > 0$

$$x = \frac{a}{a+b}\frac{m}{p} = \frac{\frac{a}{a+b} m}{p}$$

$$\underbrace{\frac{a}{a+b}}_{\in(0,1)} + \underbrace{\frac{b}{a+b}}_{\in(0,1)} = 1$$

$$\Rightarrow \boxed{px} = \frac{a}{a+b} m$$

similarly $\boxed{qy} = \frac{b}{a+b} m$

$$\Rightarrow \frac{px}{m} = \frac{a}{a+b}$$

$$\frac{qy}{m} = \frac{b}{a+b}$$

$\hookrightarrow$ income shares.

$$x = \frac{a}{a+b}\frac{m}{p} \Rightarrow \log x = \log \frac{a}{a+b} + \log m - \log p$$

Example 3 $\quad u(x,y) = Ax^a y^b \quad$ s.t. $\quad px + qy = m$

Example 4 $\quad u(x,y) \quad$ s.t. $\quad px + qy = m$

no closed form / general utility function

Section 14.1 Exercise 6 $\quad$ labor-leisure choice

Section 14.1 Exercise 9

price of good $x = p$
price of good $y = 1$

$$\max \; x^a + y \quad \text{s.t.} \quad px + y = m$$

$$\mathcal{L} = x^a + y - \lambda(px + y - m)$$

$$\mathcal{L}'_x = ax^{a-1} - \lambda p$$

$$\mathcal{L}'_y = 1 - \lambda$$

FOCs $\quad ax^{a-1} - \lambda p = 0$

$1 - \lambda = 0$

$px + y = m$

$\lambda = 1$

$$ax^{a-1} - \lambda p = 0$$

$$\Rightarrow ax^{a-1} - p = 0$$

$$\Rightarrow x^{a-1} = p/a$$

$$x = (p/a)^{\frac{1}{a-1}}$$

Solve for y:

$$px + y = m$$

$$y = m - px = m - p\left(\frac{p}{a}\right)^{\frac{1}{a-1}}$$

$=$ ___

(b) $x^* = \left(\frac{p}{a}\right)^{\frac{1}{a-1}}$ $\quad \underline{y^*} = m - p\left(\frac{p}{a}\right)^{\frac{1}{a-1}} = m - \underbrace{p^{1+\frac{1}{a-1}}\left(\frac{1}{a}\right)^{\frac{1}{a-1}}}$

$= m - p^{\frac{a}{a-1}}\left(\frac{1}{a}\right)^{\frac{1}{a-1}}$

$\frac{\partial x^*}{\partial p} = ?$ $\quad \frac{\partial y^*}{\partial p} = ?$

$\frac{\partial x^*}{\partial m} = ?$ $\quad \frac{\partial y^*}{\partial m} = ?$

$a \in (0,1)$

$$\frac{\partial x^*}{\partial p} = \underbrace{\frac{1}{a-1}}_{<0}\underbrace{\left(\frac{p}{a}\right)^{\frac{1}{a-1}-1}}_{>0}\underbrace{\left(\frac{1}{a}\right)}_{>0} < 0$$

$$\frac{\partial x^*}{\partial m} = 0$$

(c) px^* = optimal exp on x

product of two functions of p. $\quad \frac{\partial (px^*)}{\partial p} = p\frac{\partial x^*}{\partial p} + x^* \cdot 1 = \dots$

(d) Verify that $\boxed{\frac{\partial U^*}{\partial p} = \underline{-x^*(p,m)}}$

U^* = optimal / maximized utility function.

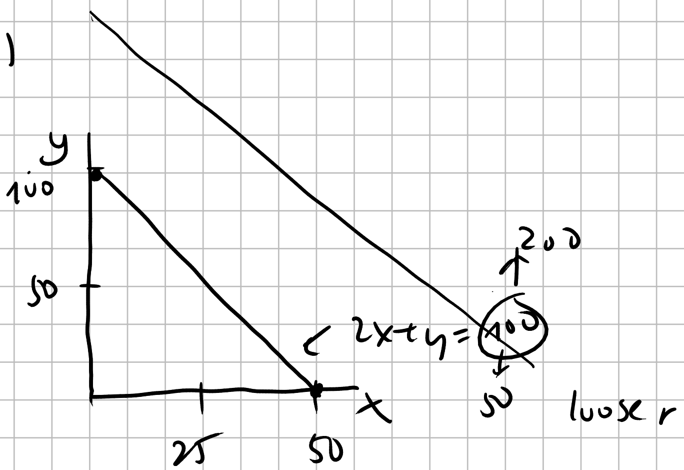
$U^* = U(x^*, y^*)$

$\downarrow$

$x^*(p, \underline{m})$

$\overset{p}{2}x + \overset{q}{1}y = 100 \to m$

$g(x,y) = c$



max/min $f(x,y)$ s.t. $g(x,y) = c$

$f < \begin{matrix} x - c \\ y - c \end{matrix}$

optimal value function $\quad f^*(c) = f(x^*(c), y^*(c))$

$$\frac{df^*(c)}{dc} = \frac{\partial f(x,y)}{\partial x}\bigg|_{x=x^*(c),\, y=y^*(c)} \frac{\partial x^*}{\partial c} +$$

$$\frac{\partial f(x,y)}{\partial y}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial y^*}{\partial c}$$

$$\overset{FOC}{=} \lambda^* \frac{\partial g(x,y)}{\partial x}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial x^*}{\partial c} + \lambda^* \frac{\partial g(x,y)}{\partial y}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial y^*}{\partial c}$$

$$= \lambda^* \Big[\underbrace{\qquad\qquad}_{=1} \Big]$$

Observe that FOC's:

$$\mathcal{L} = f(x,y) - \lambda(g(x,y) - c)$$

(I)
$$\mathcal{L}'_x = \frac{\partial f(x,y)}{\partial x} - \lambda \frac{\partial g(x,y)}{\partial x}$$
$$\mathcal{L}'_y = \frac{\partial f(x,y)}{\partial y} - \lambda \frac{\partial g(x,y)}{\partial y}$$

II
$$\frac{\partial f(x^*,y^*)}{\partial x} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial x} = 0 \Rightarrow \frac{\partial f(x,y)}{\partial x}\Big|_{x=x^*,\,y=y^*} = \lambda^* \frac{\partial g(x,y)}{\partial x}\Big|_{x=x^*,\,y=y^*}$$
$$\frac{\partial f(x^*,y^*)}{\partial y} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial y} = 0 \Rightarrow$$
$$g(x^*,y^*) = c$$

↳ Take derivative wrt c.