$$\frac{\partial f(x,y)}{\partial y}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial y^*}{\partial c}$$

$$\overset{FOC}{=} \lambda^* \frac{\partial g(x,y)}{\partial x}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial x^*}{\partial c} + \lambda^* \frac{\partial g(x,y)}{\partial y}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial y^*}{\partial c}$$

$$= \lambda^* [\underbrace{\quad\quad}_{=1}]$$

Observe that FOC's:

$$\mathcal{L} = f(x,y) - \lambda(g(x,y) - c)$$

(I)

$$\mathcal{L}_x' = \frac{\partial f(x,y)}{\partial x} - \lambda \frac{\partial g(x,y)}{\partial x}$$

$$\mathcal{L}_y' = \frac{\partial f(x,y)}{\partial y} - \lambda \frac{\partial g(x,y)}{\partial y}$$

II

$$\frac{\partial f(x^*,y^*)}{\partial x} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial x} = 0 \Rightarrow \frac{\partial f(x,y)}{\partial x}\Big|_{x=x^*,\,y=y^*} = \lambda^* \frac{\partial g(x,y)}{\partial x}\Big|_{x=x^*,\,y=y^*}$$

$$\frac{\partial f(x^*,y^*)}{\partial \cancel{y}} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial y} = 0 \Rightarrow$$

$$g(x^*,y^*) = c$$

Take derivative wrt c.

$$g(\overset{x^*(c)}{x^*}, \overset{y^*(c)}{y^*}) = c$$

Take deriv wrt c

$$g \;\begin{cases} x - c \\ y - c \end{cases}$$

$$\frac{\partial g(x,y)}{\partial x}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial x^*}{\partial c} + \frac{\partial g(x,y)}{\partial y}\Big|_{x=x^*(c),\,y=y^*(c)} \frac{\partial y^*}{\partial c} = 1$$

$$\Rightarrow \frac{df^*(c)}{dc} = \lambda^*(c)$$

$\lambda^* \to$ shadow price of the resource c (constraint constant)

$g(x,y) = c$ budget constant

Section 14.2 Example 1.

$$\max\ xy \quad \text{s.t.} \quad 2x+y=m$$

$$x^*(m)=\frac{m}{4}\ ,\quad y^*(m)=\frac{m}{2}\ ,\quad \lambda^*(m)=\frac{m}{4}$$

exact change in the optimal utility $\quad x^*(100)\,y^*(100) = 25(50)$
$= 0$

compare this to

$x^*(101)\,y^*(101)$
$= 25.25(50.5)$
$= 0$

Exercise 5

$$\alpha \ln(x-a) + \gamma \ln(y-b) = U(x,y)$$

$$\ln(x-a)^\alpha + \ln(y-b)^\beta = u(x,y)$$

$$\ln (x-a)^\alpha (y-b)^\beta = U(x,y)$$

$$(x-a)^\alpha (y-b)^\beta = \exp U(x,y)$$

Compare with Section 14.3 Example 3

Section 14.2 Exercise 3

(b) $\quad \min\ x^2+y^2 \quad \text{s.t.} \quad x+2y=\boxed{a}$ $\quad$ constraint constant

$$\mathcal{L} = x^2+y^2 - \lambda(x+2y-a)$$

$$\mathcal{L}'_x = 2x-\lambda$$

$$\mathcal{L}'_y = 2y-2\lambda$$

(x,y,λ)?

$$\begin{cases} 2x-\lambda=0 \Rightarrow \lambda=2x \\ 2y-2\lambda=0 \Rightarrow \lambda=y \\ x+2y=a \end{cases}$$

$\lambda=2x$ & $\lambda=y \Rightarrow \quad y=2x$

$y=2x$ & $x+2y=a \rightarrow \quad x+2(2x)=a$

$5x=a$

$x^*=a/5$

$\lambda^*=2a/5$

$y^*=2(a/5)=2a/5$

value function $f^*(a)$ $= (x^*)^2 + (y^*)^2 = \left(\frac{a}{5}\right)^2 + \left(\frac{2a}{5}\right)^2 = \frac{a^2}{25} + \frac{4a^2}{25} = \frac{5a^2}{25} = \frac{a^2}{5}$

$$\frac{df^*(a)}{da} = \boxed{\frac{2a}{5}} = \text{matches } \lambda^* = \frac{2a}{5} \,!$$

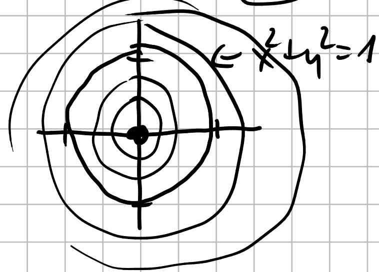
level curves of $x^2+y^2 = f(x,y)$ → plots where $x^2+y^2 =$ constant

all combinations of (x,y) which will provide the same value of f

$x^2+y^2 \neq$ negative

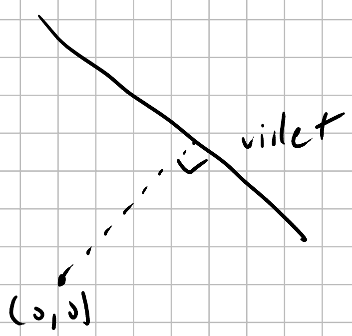
$x^2+y^2 = 0 \Rightarrow (0,0)$

$x^2+y^2 = \boxed{1} \Rightarrow (1,0), (0,1)$
$(-1,0), (0,-1)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



$x+2y = \boxed{a}$ → line.

If $a = 5 \Rightarrow x^* = \frac{a}{5} = 1, \quad y^* = \frac{2a}{5} = 2$



Recall Section 13.5 Exercise 3 / Group HW02

$g(x,y) = c$ implicit function

$x+2y = 1 \Rightarrow$ y as a function of x
$2y = 1-x$
$y = \frac{1}{2} - \frac{1}{2}x$

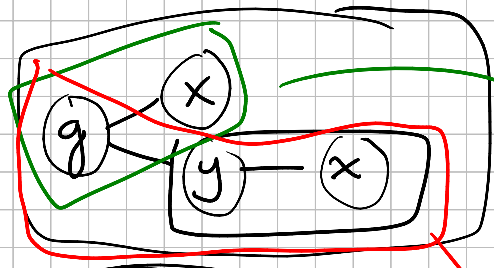
$x^2 + 2y^2 = 4 \Rightarrow$ y as a function of x

$g(x, y) = c$

$y = h(x)$

Goal: Find $\dfrac{dy}{dx}$?

Section 8.3 Example 3 already has an example "implicit differentiation"



$g(x, y) = c \Rightarrow$

Take deriv wrt x!

$\to$ constant

chain rule

$$g_1' + \overbrace{g_2' \frac{dy}{dx}} = 0$$

indirectly obtain dy/dx from

$$g_2' \frac{dy}{dx} = -g_1'$$

implicit function theorem

$$\frac{dy}{dx} = \frac{-g_1'}{g_2'} \qquad \text{Here } g_2' \neq 0$$

$$\frac{dz}{dx} = f_1' - f_2' \frac{g_1'}{g_2'} = 0$$

Let $\lambda = \dfrac{f_2'}{g_2'} \Rightarrow \quad f_1' - \lambda g_1' = 0$

$\to \lambda g_2' = f_2' \Rightarrow f_2' - \lambda g_2' = 0$

$$\lambda = \frac{f_2'}{g_2'} = \frac{f_1'}{g_1'} \Rightarrow -\frac{f_1'}{f_2'} = -\frac{g_1'}{g_2'}$$