Section 13.5 Exercise 3

$\max \quad 9x+8y-6(x+y)^2$

subject to $0 \le x \le 4$, $0 \le y \le 3$, $-x+2y \le 2$

$x \ge 0$ and $x \le 4$

$y \ge 0$ and $y \le 3$

$\Rightarrow$ Convert into a form similar to Section 14.8 (with 14.9, 14.10)

$\max f(x,y)$

s.t. $g(x,y) \le c$

$\max \quad 9x+8y-6(x+y)^2$

subject to
$$\begin{cases} -x \le 0 \\ x \le 4 \\ -y \le 0 \\ y \le 3 \\ -x+2y \le 2 \end{cases}$$
in the "right" format

KT (Kuhn-Tucker conditions)

KKT (Karush-Kuhn-Tucker conditions)

λ in new recipe is always ≥ 0

in contrast, you have seen situations before Section 14.8 where λ could be negative (Section 14.3 Example 1)

$\max \quad ax+by$

subject to $\alpha x^2 + \beta y^2 \le L$

$\mathcal{L} = ax+by - \lambda(\alpha x^2+\beta y^2 - L)$

$\mathcal{L}'_x = a - 2\lambda\alpha x = 0 \qquad \mathcal{L}''_{xx} = -2\lambda\alpha < 0$

$\mathcal{L}'_y = b - 2\lambda\beta y = 0 \qquad \mathcal{L}''_{yy} = -2\lambda\beta < 0$

$\mathcal{L}''_{xy} = 0$

KKT conditions:

$$\left.\begin{aligned} a - 2\lambda\alpha x^* &= 0 \\ b - 2\lambda\beta y^* &= 0 \end{aligned}\right\} \text{ stationary pts of Lagrangian}$$

$\lambda \ge 0$ and $\lambda = 0$ if $\alpha(x^*)^2 + \beta(y^*)^2 < L$ Complementary slackness

$\alpha(x^*)^2 + \beta(y^*)^2 \le L \rightarrow$ constraint must be satisfied

Find (x^*, y^*, λ) so that all KKT conditions are satisfied.

Observe that x^*, y^*, λ have to be positive. Why?

$$a - 2\lambda\alpha x^* = 0 \Rightarrow \underbrace{a}_{>0} = 2\lambda \underbrace{\alpha}_{>0} x^*$$

either λ, x^* are both positive

$\boxed{\text{or } \lambda, x^* \text{ are both negative}}$

we can rule out!

either because $\lambda \geq 0$ by C-S condition

or you use the econ. context (cannot produce negative units)

Similarly, $b - 2\lambda\beta y^* = 0$

$$\Rightarrow \underbrace{b}_{>0} = 2\lambda \underbrace{\beta}_{>0} y^*$$

$$a = 2\lambda\alpha x^* \Rightarrow \frac{a}{2x^*\alpha} = \lambda$$

$$b = 2\lambda\beta y^* \Rightarrow \frac{b}{2y^*\beta} = \lambda$$

$$\frac{a}{2x^*\alpha} = \frac{b}{2y^*\beta}$$

Because $\lambda > 0 \Rightarrow \alpha(x^*)^2 + \beta(y^*)^2 = L$

C-S: if $\underbrace{\alpha(x^*)^2 + \beta(y^*)^2 < L}_{\text{not true}} \Rightarrow \cancel{\lambda = 0}$

$$a y^* \beta = b x^* \alpha \Rightarrow y^* = \frac{b\alpha}{a\beta} x^*$$

Subst into $\alpha(x^*)^2 + \beta(y^*)^2 = L$

$$\alpha(x^*)^2 + \beta \frac{b^2\alpha^2}{a^2\beta^2}(x^*)^2 = L$$

$$\left(\alpha + \frac{b^2\alpha^2}{a^2\beta}\right)(x^*)^2 = L$$

$$(x^*)^2 = \left(\alpha + \frac{b^2\alpha^2}{a^2\beta}\right)^{-1} L$$

$$x^* = \underbrace{\pm}_{\text{rule out}} \sqrt{\left(\alpha + \frac{b^2\alpha^2}{a^2\beta}\right)^{-1} L}$$

$y^* =$ ___

$\lambda =$ ___

$$\alpha(x^*)^2 + \beta(y^*)^2 = L$$

Constraint was satisfied at equality! Constraint is binding/active.

If you say that $\alpha(x^*)^2 + \beta(y^*)^2 < L$,

it means that $\underbrace{\alpha(x^*)^2 + \beta(y^*)^2}_{\text{labor actually used}} + \underbrace{\text{unused labor}}_{\text{slack}} = \underbrace{L}_{\text{total available labor.}}$

Section 14.8 Example 2

$2x - 2\lambda x = 0 \Rightarrow 2x(1-\lambda) = 0$

$\boxed{2y + 1 - 2\lambda y = 0} \Rightarrow \boxed{2y(1-\lambda) = -1} \Rightarrow \lambda \neq 1$ Why?

$\lambda \geq 0$ and $\lambda = 0$ if $\underline{x^2+y^2<1}$

$\boxed{x^2+y^2 \leq 1}$

Because if it were the case that $\lambda = 1$
$\Rightarrow 0 = -1$
This is impossible!

Because $\lambda \neq 1$, $\overset{\neq 0}{\boxed{2}}\, x \overset{\neq 0}{(1-\lambda)} = 0 \Rightarrow x = 0$

$x=0$!

Case 1 $\underline{x^2+y^2=1}$

$0^2 + y^2 = 1$

$y = \pm 1$

Case 1a $y = 1$

$\boxed{\text{By C-S, } \lambda \geq 0.}$

$2y + 1 - 2\lambda y = 0$

$2(1) + 1 - 2\lambda(1) = 0$

$3 - 2\lambda = 0$

$3 = 2\lambda$

$3/2 = \lambda$

A solution candidate: $x=0, y=1, \lambda = \frac{3}{2}$

Case 1b $y = -1$

By C-S, $\lambda \geq 0$

$2y + 1 - 2\lambda y = 0$

$2(-1) + 1 - 2\lambda(-1) = 0$

$-1 + 2\lambda = 0$

$-1 = -2\lambda$

$1/2 = \lambda$

A solution candidate: $x=0, y=-1, \lambda = \frac{1}{2}$

Case 2 $x^2+y^2<1$

By C-S, $\lambda = 0$.

$2y + 1 - 2\lambda y = 0$

$2y + 1 - 2(0)y = 0$

$2y + 1 = 0$

$y = -1/2$

A solution candidate: $x=0, y=-1/2, \lambda = 0$

$\cancel{f(0,1) = 1} \Rightarrow (0,1)$ maximizes $f(x,y) = x^2+y^2+y-1$ s.t. $x^2+y^2 \leq 1$

$f(0,-1) = -1$

$f(0,-\frac{1}{2}) = -5/4$ minimizes $f(x,y)$ s.t. $x^2+y^2 \leq 1$

$f(x,y) = x^2 + y^2 + y - 1$

$$x^2 + y^2 + y - 1 = -\frac{5}{4}$$

$$\Rightarrow x^2 + \left(y^2 + y + \frac{1}{4}\right) - 1 - \frac{1}{4} + \frac{5}{4} = 0$$

$$\Rightarrow x^2 + \left(y + \frac{1}{2}\right)^2 = 0$$

$$(x-0)^2 + \left(y - \left(-\frac{1}{2}\right)\right)^2 = 0$$

equation of a circle

$$x^2 + y^2 + y - 1 = 1$$

$$\Rightarrow x^2 + \left(y^2 + y + \frac{1}{4}\right) = 1 + 1 + \frac{1}{4}$$

$$\Rightarrow x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{9}{4}$$

$$(x-0)^2 + \left(y - \left(-\frac{1}{2}\right)\right)^2 = 9/4$$

radius $r = 3/2$

Lagrangian is concave/convex $\rightarrow$ Ch 13

Example 2 $\mathcal{L} = x^2 + y^2 + y - 1 - \lambda(x^2 + y^2 - 1)$

$\mathcal{L}'_x = 2x - 2\lambda x$

$\mathcal{L}'_y = 2y + 1 - 2\lambda y$

$\mathcal{L}''_{xx} = 2 - 2\lambda$

$\mathcal{L}''_{yy} = 2 - 2\lambda$

$\mathcal{L}''_{xy} = 0$

Case 1a $\lambda = \frac{3}{2}$	Case 1b $\lambda = \frac{1}{2}$	Case 2 $\lambda = 0$
$2 - 2\left(\frac{3}{2}\right) = -1 < 0$	$2 - 2\left(\frac{1}{2}\right) = 1 > 0$	$2 > 0$
$2 - 2\left(\frac{3}{2}\right) = -1 < 0$	$2 - 2\left(\frac{1}{2}\right) = 1 > 0$	$2 > 0$
$\mathcal{L}''_{xx}\mathcal{L}''_{yy} - (\mathcal{L}''_{xy})^2 = 1 > 0$	$\mathcal{L}''_{xx}\mathcal{L}''_{yy} - (\mathcal{L}''_{xy})^2 = 1 \cdot 1 - 0 = 1 > 0$	$\mathcal{L}''_{xx}\mathcal{L}''_{yy} - (\mathcal{L}''_{xy})^2 = 2(2) - 0^2 = 4 > 0$
max	(min)	min
	$f(0,-1) = -1$ (rule out)	$f\left(0, -\frac{1}{2}\right) = -\frac{5}{4}$

For Example 2, you can use the Extreme Value Thm

$x^2 + y^2 \leq 1 \rightarrow$ closed & bounded set

$f(x,y) \rightarrow$ continuous.